THE METHODS OF CALCULATING WEIGHTS OF HEIGHT DIFFERENCES

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Abstract: *Solving various geodetic tasks begins with defining and actualising the mathematical basis. It is reflected in the development of different types of geodetic networks - height, position, and three-dimensional ones. Some networks rely on the existing ones, and some are developed as independent geodetic networks. Measurement and processing of measurement results in these networks are carried out in both cases. Processing, among other things, involves adjustment of the measured values. Height geodetic networks are most often developed using the geometric levelling method. The accuracy of the geometric levelling (besides the accuracy of the used level) depends on the line length, the number of stations and the height difference between the benchmarks. Therefore, different methods of calculating weights can be used in adjustment of height networks. This paper will analyse the results of adjustment of one height network with differently calculated weights in order to determine whether the method of calculating weights has an effect on the final values - the height of points.*

Keywords: *adjustment, levelling network, measurement weights*

1. INTRODUCTION

The greatest deficiency of the height reference networks is the time period in which they were developed, and thus the used methods also changed. Works on the creation of the height reference network in the territory of the former SFR Yugoslavia started in 1871 and, with interruptions, lasted until 1973 (1986). Stabilised points (benchmarks) are

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subject to various geodynamic processes caused by the displacement of tectonic plates or local disturbances of the Earth’s surface.

Until the Second World War, our country had regulations only related to surveying details in towns. Due to the poor outside conditions under which observations were performed in towns, which resulted in the reduction of accuracy of the measurement results, the town networks were of poor quality and accuracy compared to those out of towns. For that reason, the "Rulebook on the State Survey II-A Part - Basic Geodetic Works in Towns" was adopted in 1956. This Rulebook precisely determines the process of developing networks, measurement methods in the networks, and the way of their adjustment aimed at meeting the most demanding requirements of town development.

At the beginning, according to the Rulebook, the height networks were adjusted in different ways - a basic network according to the conditional measurement method, a filling network depending on how it was developed (it was only necessary to be adjusted as a whole), and the network of inserted traverses according to the known procedure (adjustment of individual traverses). The development of technology has successfully solved the problem of network adjustment, i.e. networks are adjusted according to the parametric adjustment and without division into the basic and filling network. The best relative relation of points within a network is achieved when the network is adjusted as a free network in the local coordinate system.

For the needs of engineering works, special-purpose networks have been developed. The measurement methodology is almost identical as with town networks (the difference is in the networks for works of less accuracy when the precise levelling method is not applied), and for the adjustment, the completely same method is applied - parametric adjustment.

It should be noted that the parametric adjustment method is applied in the previous accuracy assessment of the quality of geodetic networks (assessment of accuracy and reliability of networks during the development of the network project).

In the method of measuring height differences by geometric levelling, the accuracy of the measurement depends on the length of the level line, the number of stations and the height difference between the benchmarks [1]. It means that different criteria can be applied when determining the weight of the measured values. Most often, weights are determined in proportion to the length of the level line (the Rulebook’s provisions), and the calculation of weights according to the number of stations between the benchmarks can be applied as well. Determining weights according to the previous accuracy assessment of the height differences measurements involves the calculation of the influence of all the values that affect the accuracy of the measurement. This method of calculating weights began to be applied with the development of computer technology since it is assumed that for each measured height difference, a mean measurement error dependent on three values must be calculated (the tendency was to have as few calculations as possible).

This paper will present the influence the methods of calculating weights of height differences has on the adjustment results - heights of benchmarks. The influence will be presented in a concrete case – a levelling network.

2. MATHEMATICAL MODELS

The mathematical model used for geodetic networks adjustment, which best describes the actual state of such a system, is the Gauss-Markov Model (GMM) [2], which uses the least squares method to estimate the parameters. The Gauss-Markov Model consists of a functional and stochastic part that can be presented mathematically:

- Linear functional model:

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{f} \tag{1}$$

- Stochastic model:

$$\mathbf{K}_l = \sigma_0^2 \mathbf{Q}_l \tag{2}$$

The basic assumption of the measurements under which the least squares method is applied is that the mathematical expectation of measurement errors is zero, i.e. that measurement errors follow a normal distribution with a mathematical expectation of zero and a covariance matrix **K**.

$$M(\varepsilon) = M(v) = 0, \varepsilon \sim \mathbf{N}(0, \mathbf{K}) \tag{3}$$

The measured values are random values that follow the normal probability distribution expressed in the form:

$$\mathbf{l} \sim \mathbf{N}(\boldsymbol{\mu}_1, \mathbf{K}_l) \tag{4}$$

Where $\boldsymbol{\mu}_1$ is the expectation vector and $\mathbf{K}_l$ is the covariance matrix of the measured values.

2.1. Functional model

During parametric adjustment by a functional model, a functional connection between the measured values **l** and unknown parameters **X** is defined. In the general case, the link functions are nonlinear and in the vector form they are:

$$\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v} = \mathbf{F}(\hat{\mathbf{X}}) \tag{5}$$

where the vectors are:

$$\mathbf{l} = \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \\ \vdots \\ \Delta h_n \end{pmatrix}, \ \hat{\mathbf{l}} = \begin{pmatrix} \Delta \hat{h}_1 \\ \Delta \hat{h}_2 \\ \vdots \\ \Delta \hat{h}_n \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_{\Delta h_1} \\ v_{\Delta h_2} \\ \vdots \\ v_{\Delta h_n} \end{pmatrix}, \ \mathbf{F}(\hat{\mathbf{X}}) = \begin{pmatrix} F_1(\hat{X}) \\ F_2(\hat{X}) \\ \vdots \\ F_n(\hat{X}) \end{pmatrix} \tag{6}$$

$\mathbf{l}$ - vector of measured values,
$\hat{\mathbf{l}}$ - vector of adjusted (estimated) values,

v - vector of corrections,
$\mathbf{F}\left(\hat{\mathbf{X}}\right)$ - vector of nonlinear, mathematical functions, and
$\hat{\mathbf{X}}$ - vector of adjusted (estimated) parameters.

The least squares method implies the linearity of the link function. This is achieved by expanding the function to Taylor series around approximate values:

$$l_i + v_i = F_i\left(H_{1_0}, H_{2_0}, \ldots, H_{n_0}\right) + \frac{\partial F_i}{\partial H_{1_0}} dH_1 + \frac{\partial F_i}{\partial H_{2_0}} dH_2 + \cdots + \frac{\partial F_i}{\partial H_{n_0}} dH_n \quad \left(i = 1, 2, \ldots, n\right) \tag{7}$$

Partial derivatives of the function by unknown parameters are:

$$a_i = \frac{\partial F_i}{\partial H_{1_0}} dH_1,\; b_i = \frac{\partial F_i}{\partial H_{2_0}} dH_2,\; \ldots,\; u_i = \frac{\partial F_i}{\partial H_{n_0}} dH_n \;,\; \left(i = 1, 2, \ldots, n\right) \tag{8}$$

Free terms:

$$f_i = F_i\left(H_{1_0}, H_{2_0}, \ldots, H_{n_0}\right) - l_i \,,\quad \left(i = 1, 2, \ldots, n\right) \tag{9}$$

Correction equations are:

$$v_i = a_i dH_1 + b_i dH_2 + \cdots + u_i dH_n + f_i \tag{10}$$

Correction equations in the matrix form:

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{f} \tag{11}$$

Or

$$\begin{pmatrix} v_{\Delta h_1} \\ v_{\Delta h_2} \\ \vdots \\ v_{\Delta h_n} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & \cdots & u_1 \\ a_2 & b_2 & \cdots & u_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & b_n & \cdots & u_n \end{pmatrix} \begin{pmatrix} dH_1 \\ dH_2 \\ \vdots \\ dH_n \end{pmatrix} + \begin{pmatrix} f_{\Delta h_1} \\ f_{\Delta h_2} \\ \vdots \\ f_{\Delta h_n} \end{pmatrix} \tag{12}$$

A - design matrix,
f - vector of free terms.

2.2. Stochastic model

The stochastic model of parametric adjustment refers to the vector of measured values **l**. When the measured values are stochastically dependent, the covariance matrix $\mathbf{K}_l$ or the cofactor matrix $\mathbf{Q}_l$ should be used:

$$\mathbf{K}_l = \sigma_0^2 \mathbf{Q}_l \tag{13}$$

where σ_0 is the standard deviation of the weight unit of adjusted values. When the measured values are stochastically independent, then all the elements outside the main diagonal of the matrix $\mathbf{K}_l$ are equal to zero:

$$\mathbf{K}_l = \begin{pmatrix} \sigma_{\Delta h_1}^2 & & & \\ & \sigma_{\Delta h_2}^2 & & \\ & & \ddots & \\ & & & \sigma_{\Delta h_n}^2 \end{pmatrix} = \sigma_0^2 \cdot \begin{pmatrix} \frac{1}{p_1} & & & \\ & \frac{1}{p_2} & & \\ & & \ddots & \\ & & & \frac{1}{p_n} \end{pmatrix} = \sigma_0^2 \mathbf{P}_l^{-1} \tag{14}$$

where:

$$\mathbf{P}_l = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix} = Diag\left(p_1\ p_2\ \dots\ p_n\right) \tag{15}$$

2.3. Adjustment algorithm

The matrix **A** (design matrix) is defined based on performed observations, i.e. based on how the benchmarks are interconnected. The weight matrix of measured values is diagonal, and its diagonal terms can be calculated in different ways. The first method is based on the mean error of the measured height difference, which is calculated according to the formulas for the previous measurement accuracy assessment. Weight is calculated by the expression:

$$p_i = \frac{1}{\sigma_{\Delta h_{POT}}^2} \tag{16}$$

where the mean error of the measured height difference is calculated from the expression for the previous accuracy assessment of the height difference between two benchmarks at a distance of 1 km [1]:

$$\sigma_{\Delta h}^2 = \frac{n_h}{2n}\left[\sigma_{RM}^2 + 2\sigma_z^2 + \frac{d^2}{\rho^{"2}}\left(2\sigma_r^2 + \sigma_O^2\right)\right] + \Delta h^2\left(\sigma_{ML}^2 + \gamma_p^2(t)\cdot\Delta t^2\right) \tag{17}$$

where:

- d - average length of the line of sight from the level to the rod,
- σ_{RM} - instrument standard error,
- σ_r - mean refraction error,
- σ_O - mean error in rod reading,

σ_{ML} - standard error of mean meter rods pair,
γ_p - temperature coefficient of expansion of invar tape,
Δt - temperature difference during calibration of rods and observation,
σ_z - mean error of rounded readings,
Δh - height difference between two benchmarks at the distance of 1 km,
n - number of measurements (for forward-backward $n = 1$),
n_h - number of stations per 1 km.

For certain sources of errors, general values [1] are used, determined from a large number of measurements, and some determined from measurements made under specific conditions can also be used (for example, the mean error in rod reading).
Another method of calculating weights is from the length of the levelling line:

$$p_i = \frac{1}{s} \tag{18}$$

where s is the total length between the benchmarks (mean from forward-backward). The last method of calculating weights is based on the number of stations between the benchmarks (mean from forward-backward):

$$p_i = \frac{1}{n_h} \tag{19}$$

After matrixes **A** and **P** are formed, the matrix of normal equations is obtained:

$$\mathbf{N} = \mathbf{A}^T\mathbf{P}\mathbf{A} \tag{20}$$

The cofactor matrix of unknown values $\mathbf{Q}_h$ is obtained by expanding the matrix **N** with the data criteria matrix $\mathbf{H}_d^T$ and its inversion:

$$\mathbf{H}_d^T = [...0 \ 0 \ 1 \ 0...] \tag{21}$$

$$\left(\mathbf{N} + \mathbf{H}_d\mathbf{H}_d^T\right)^{-1} = \begin{vmatrix} \mathbf{N} & \mathbf{H}_d \\ \mathbf{H}_d^T & 0 \end{vmatrix} \tag{22}$$

$$\mathbf{Q}_{\hat{x}} = \left(\mathbf{N} + \mathbf{H}_d\mathbf{H}_d^T\right)^{-1}\mathbf{N}\left(\mathbf{N} + \mathbf{H}_d\mathbf{H}_d^T\right)^{-1} \tag{23}$$

Based on the matrix $\mathbf{Q}_h$ the following matrices are calculated:

- Cofactor matrix of estimated measurements:

$$\mathbf{Q}_{\hat{l}} = \mathbf{A}\mathbf{Q}_{\hat{x}}\mathbf{A}^T \tag{24}$$

- Cofactor matrix of estimated residuals:

$$\mathbf{Q}_v = \mathbf{P}^{-1} - \mathbf{Q}_{\hat{l}} \tag{25}$$

- Reliability matrix:

$$\mathbf{R} = \mathbf{Q}_v \mathbf{P} = \left(\mathbf{P}^{-1} - \mathbf{Q}_{\hat{l}}\right)\mathbf{P} = \left(\mathbf{P}^{-1} - \mathbf{A}\mathbf{Q}_{\hat{x}}\mathbf{A}^T\right)\mathbf{P} \tag{26}$$

Local measures of internal reliability r_{ii} are on the diagonal **R**. The coefficient r_{ii} is an indicator of the possibility of network geometry to detect the impacts of gross errors in the corrections of measurement results. It defines the percentage of gross errors in the measurement included in the correction after adjustment.
Unknown values, heights of benchmarks, are calculated through the expression:

$$\hat{\mathbf{x}} = -\mathbf{N}^{-1}\mathbf{n} = -\mathbf{Q}_x \mathbf{n} \tag{27}$$

where **n** is a vector of free terms of normal equation:

$$\mathbf{n} = \mathbf{A}^T \mathbf{P}_l \mathbf{f} \tag{28}$$

and **f** is a vector of free terms. Assessing the accuracy of unknown parameters, heights, is done through the expression:

$$\sigma_{x_i} = \sigma_0 \sqrt{\mathbf{Q}_{x_i x_i}} \tag{29}$$

3. EXPERIMENTS AND RESULTS

3.1. Description of the network, measurement and processing

The main characteristics of the town levelling network described in this paper are:

- number of benchmarks: 103,
- number of polygons: 15,
- total network length: 25 km,
- length of the levelling line: min. 90 m, max. 450 m, average 208 m,
- height differences: min. -24,8 m, max. 33,5 m, average -0,3 m,
- polygon perimeter: min. 1.70 km, max. 3.78 km, average 2.42 km,
- mean network height 41,7 m.

The measurements in the network were carried out using the method of geometric levelling, levelling from the middle. The measurements were made in both directions, one direction in the morning and the other in the afternoon. The measurements were performed using the Zeiss Koni007 instrument, with the declared levelling accuracy at 1 km in both directions of $0{,}7\ \mathrm{mm}/\sqrt{\mathrm{km}}$.

Processing of the measurement results of the town levelling network was made in the system of normal heights. The network date was positioned with the existing benchmark of the precision levelling with a height of 11.4 m.

3.2. Adjustment results

The network was adjusted by the parametric adjustment in three variants. In each variant, the measurement results were the same, and only weights were different. In the variant 1, weights were calculated according to expression (16) - previous accuracy assessment, in the variant 2 according to the expression (18) - lengths of lines and in the variant 3 according to expression (19) – the number of stations. The basic indicators of adjustment are shown in Table 1.

After adjustment, the differences of adjusted heights were calculated. The characteristic values are shown in Table 2.

Figures 1 and 2 show graphs of differences of adjusted heights, depending on the heights of the benchmarks. Based on Table 2, it can be concluded that the method of calculating weights has an effect on the adjusted values of adjustment since the differences in the adjusted values are greater than the height accuracy. Moreover, Figures 1 and 2 show that there is a certain linear dependence of the differences on the height of the benchmarks. This trend is more pronounced with the height differences when weights are calculated according to the previous accuracy assessment and according to the lengths of the lines (in this case, the values of differences are also greater).

Table 1 Adjustment by variants [mm]

Value	Variant 1	Variant 2	Variant 3
σ_0	0,044	0,024	0,003
σ_H - max.	2,42	2,98	2,50
σ_H - min.	0,58	1,05	0,72
σ_H - aver.	1,60	2,31	1,82
$\sigma_{\Delta h}$ - max.	1,74	1,41	1,39
$\sigma_{\Delta h}$ - min.	0,37	0,69	0,46
$\sigma_{\Delta h}$ - aver.	0,84	1,08	0,91

Table 2 Differences of adjusted heights [mm]

Value	V. 1 - V. 2	V. 1 - V. 3
R - max.	14,21	3,35
R - min.	-7,19	-5,41
R - aver.	-1,32	-0,69

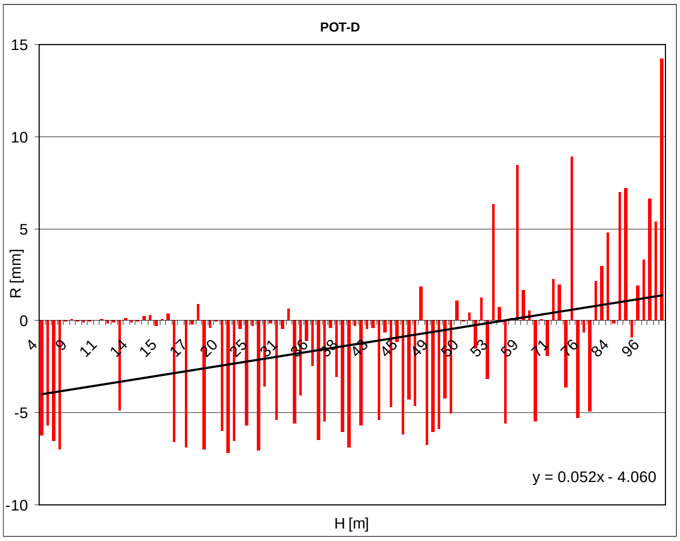

Figure 1 Height differences Variant 1 - Variant 2

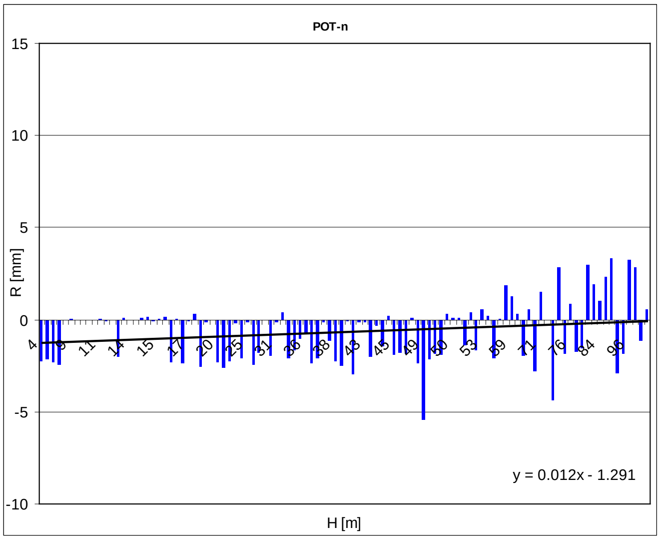

Figure 2 Height differences Variant 1 - Variant 3

4. CONCLUSION

In order to verify the effect of methods of calculating weights while adjusting the height network measured by the method of the geometric levelling, adjustment of one town height network was performed in three variants. In the variant 1, weights were calculated according to the previous accuracy assessment, in the variant 2 according to lengths of lines and in the variant 3 according to the number of stations. With the development of technology and computers, the first variant is most often applied. The variant 2, weights proportional to lengths, was applied in accordance with the Rulebook on the State Survey because in the Rulebook, the error of measuring the height difference was calculated only depending on the length of the line. The variant 3, according to the number of stations, is the least commonly used variant for calculating weights.

By comparing the adjustment results, point heights, the following conclusions can be drawn:

- the method of calculating weights has an effect on the value of the adjusted values because the differences are greater than the errors of the adjusted values differences,
- there is a certain linear dependence of the differences on the heights of the benchmarks,
- the biggest differences among the models are when weights are calculated in proportion to lengths of other two models as well, indicating that the method of calculating weights (and mean measurement errors), according to the Rulebook does not include all sources of errors that may occur during the measurement (this has been confirmed several times during measurements in different levelling networks while calculating the permitted deviation of the forward-backward measurement).

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NAČINI RAČUNANJA TEŽINA VISINSKIH RAZLIKA

Apstrakt: *Rešavanje različitih geodetaskih zadataka počinje definisanjem i realizalizacijom matematičke osnove. To se ogleda u razvijanju različitih vrsta geodetskih mreža - visinske, položajne, trodimenzionalne. Neke mreže se oslanjaju na već postojeće, a neke se razvijaju kao samostalne geodetske mreže. U oba slučaja se vrši merenje i obrada rezultata merenja u tim mrežama. Obrada, između ostalog, podrazumeva i izravnanje merenih veličina. Visinske geodetske mreže se najčešće razvijaju korišćenjem metode geometrijskog nivelmana. Tačnost geometrijskog nivelmana (osim tačnosti korišćrnog nivelira) zavisi od dužine linije, broja stanica i visinske razlike između repera. Zato se kod izravnanja visinskih mreža mogu koristiti različiti načini računanja težina. U ovom radu će biti analizirani rezultati izravnanja jedne visinske mreže sa različito sračunatim težinama kako bi se utvrdilo da li način računanja težina ima uticaj na konačne vrednosti - visine tačaka.*

Ključne reči: *Izravnanje, nivelmanska mreža, težine merenja*