INFLUENCE OF WEIGHTS ON THE UNKNOWNS' ESTIMATION IN 1D GEODETIC NETWORKS

Žarko Nestorović[1*], Novica Trifković[2]

[1] JSC EPS Belgrade, Belgrade, Serbia
[2] University of Novi Sad, Faculty of Civil Engineering Subotica, Serbia
[*] corresponding author: zarko.nestorovic@eps.rs

Paper type: Original scientific paper
Received: 2024-11-18
Accepted: 2025-12-10
Published: 2024-12-29

UDK: 528.41
DOI: 10.14415/JFCE-910

ABSTRACT

This paper represents the results of provided research about influence of weights on the unknowns. The research was conducted in the cases of classical Least Square estimation of unknows as well as the generalized matrix inversion. The research was provided for the six cases of 1D networks starting from the simplest cases to the more complicated. The research encompassed hundred random variations of weights values per each researched case in the range of (-0,3 - +0,3) related to the starting weights value. The results showed relatively small influences of weights in considered range on the variations of the unknowns. The possible practical importance of this research is that the influence of wrong determined weights could be modelled and estimated in advance.

KEYWORDS:

LEAST SQUARE METHOD, GENERALIZED MATRIX INVERSION, STATISTICAL ANALYSIS, SENSITIVITY, STATISTICAL SIGNIFICANCE

1 INTRODUCTION

The question of the influence of weights is quite interesting in the literature of geodetic networks analysis. This question arises from the certain fundamental characteristics of the geodetic network adjustment. The estimation of unknown heights or coordinates in the geodetic networks are dependent on the measurements' errors distribution over the field of measurements but also are dependent on the reliability of every single measurement i.e. of its weight. The weight of every single measurement is defined by its standard or mean square error and calculated as inverse value of mean square error value. Another assumption is that errors in the field of measurements are not correlated i.e. that weight matrix is diagonal. This assumption is also accepted in this research.

The investigation of levelling (1D) networks was conducted by utilizing robust analysis [1] where their limitations were stressed. The uncertainty assessment in geodetic network was the issue of analysis by utilizing the Monte-Carlo-simulations [6].

The weights of height differences calculation are explained in different books which deals with adjustment of geodetic networks [7, 8].

The more detailed analysis of height differences weights calculation is provided in [2] This research was provided on the real 1D (levelling) network with the large number of benchmarks and relatively large total length having the purpose to identify the influence of weights calculated in three different ways. The conclusion was that method of weights calculation has the effects on the adjusted values of unknown parameters (in this case: benchmarks' heights). This conclusion are important because the leveling networks are considered as highly accurate and it is of crucial importance to avoid negative influences of weights in the process of adjustment. In other words: every effort should be done in order to avoid the reduction of unknown parameters' accuracy in the proces of adjustment.

The main purpose of this research is to find out how the variation of weights affect the unknown height of benchmarks for the 1D (levelling) network. In this sense the variations in weight values can be considered as the errors in their determination i.e. the deviations from their true values.

The research was conducted on the theoretical base (simulation) by forming the one hundred of randomly generated weights for each of six levelling networks starting from the simplest model to the more complex. Complexity of model was increased with the increase of benchmark number as well as the increase the number of measured height differences. The differences were adopted to be the same for each network and the results showed the scalability of the influence of variated weights on the unknown's values. The research was provided for both classical Least Squares Method (LSM) and for generalized solution.

2 MATERIALS AND METHODS

Materials for this research were obtained by forming the starting the weight matrix as an identity matrix and after that creating its increment on the random base in the interval of (-0,3; +0,3).

Starting from the well-known equation for unknowns' value estimation by utilizing the LSM (classical or generalized solution) [4]:

$$\hat{x} = -(A^T P A)^{-1} A^T P f = -Q_x A^T P f \tag{1}$$

$$\hat{x} = -(A^T P A)^{+} A^T P f = -\, Q_x^{+} A^T P f \tag{2}$$

it immediately follows that changes in the weights or its wrong determination should cause estimated values change. This could be explicated as follows:

$$\hat{x} + \delta x = -[A^T (P + \Delta P) A]^{-1} A^T (P + \Delta P) f \tag{3}$$

$$\hat{x} + \delta x = -[A^T (P + \Delta P) A]^{+} A^T (P + \Delta P) f \tag{4}$$

$$\hat{x} + \delta x = -[A^T P A + A^T \Delta P A]^{-1} [A^T P + A^T \Delta P] f \tag{5}$$

$$\hat{x} + \delta x = -[A^T P A + A^T \Delta P A]^{+} A^T (A^T P + A^T \Delta P) f \tag{6}$$

where:

- A: design matrix of the levelling network
- P: weight matrix
- ΔP: increment of weight matrix
- f: vector of free terms
- Q_x: cofactor matrix for classical adjustments (inverse matrix)
- Q_x^{+}: cofactor matrix for free network adjustments (pseudoinverse matrix) and
- $\hat{x}$: estimated value of unknows.

From the above formulas it is obvious that exact solution of this problem is a quite complex task. Even for the non-singular matrix it requires significant calculations. According to [5] the solution for sum of two matrixes is as follows:

$$(G + H)^{-1} = G^{-1} - \frac{1}{1+g} G^{-1} H G^{-1} \tag{7}$$

$$g = \mathrm{tr} H G^{-1} \tag{8}$$

where:

- G, H: two regular and rectangular matrixes and
- g: trace of matrixes production.

For the case of generalized inverse of sum of two matrices, the calculation is more complicated (see for example [3]).

The complexity of this problem directed the research on its simplification and only to forming the matrix ΔP and calculating the matrix P in following way:

$$P_j = \mathrm{diag}\,[p_1 + \Delta p_1 \; p_2 + \Delta p_2 \ldots \; p_m + \Delta p_m] \tag{9}$$

$$\mathrm{P}_j = \mathrm{diag}\,[1 + \Delta p_1 \; 1 + \Delta p_2 \, \ldots \; 1 + \Delta p_m] \tag{10}$$

where:

- P_j: the matrix formed in j^{th} - iteration ($j = 1, 2, \ldots, 100$);
- when $j = 1$ then $\mathrm{P}_1 = \mathrm{I}$ (identity matrix) and
- Δp_i: increment for weight i.

Trying to avoid the theoretical and practical issues the increment of each weight was calculated as follows:

$$\Delta p_i = \mathrm{rand}(x) \tag{11}$$

where $\mathrm{rand}(x)$ denotes a random function and the calculation is repeated until the condition $\Delta p_i \in [-0{,}3;\ +0{,}3]$ was fulfilled for each measurement. This means that the variation of weights was allowed to the ±30% of the starting values when the weights were mutually equal and consequently equals to unity. For the purpose of this research six different designs of 1D levelling networks were considered. The levelling networks were designed from the simplest consisted of three benchmarks to the one consisted of twelve benchmarks. The plan of height differences measurements for each network is given on the Figure 1.

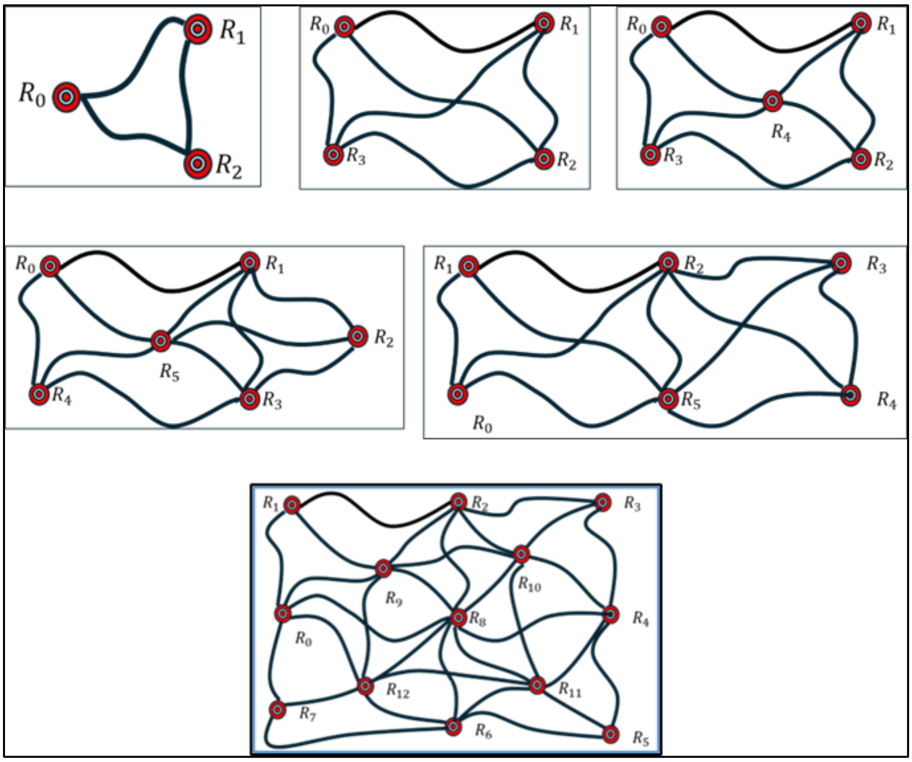

Figure 1. The 1D network shapes which were used for research (red points are the positions of benchmarks and lines are the directions of levelling)

The value for vector of free terms $\mathbf{f}$ was as follows:

$$\mathrm{f}^T = [1 \quad \cdots \quad 1] \tag{12}$$

Choosing these values for the vector f has the consequence in obtaining the sum of product in formulas (3) and (4) represents the values which are dependent only on the influence of weights change:

$$\hat{x}_i = -\sum_{k=1}^{m} f_k \sum_{\substack{i=1\\j=1\\k=1}}^{\substack{n\\m\\n}} q_{i,k} a_{k,j} p_j \tag{13}$$

where:

- f_k: the kth term of vector of measurements;
- n: number of unknowns;
- m: number of measurements and
- $q_{i,k}, a_{k,j}$ and p_j: the elements of matrices $\mathrm{Q_x}, \mathrm{A^T}$ **and** P respectively.

Difference between values of unknowns obtained for the different values of weight matrix represents the influence of weights change as follows:

$$dx_{i,j} = \hat{x}_j - \hat{x}_i \tag{14}$$

where $\hat{x}_j$ and $\hat{x}_i$ are the estimated values for the certain unknown x obtained for weight matrix j and i, respectively.

The accuracy of estimated unknown's value is obtained as follows:

$$m_{x_i} = m_0\sqrt{q_{x_i}} \tag{15}$$

Accepting $m_0 = 1$ for each network and combination leads to:

$$m_{x_i} = \sqrt{q_{x_i}} \tag{16}$$

and consequently

$$m_{dx_{i,j}} = \sqrt{q_{x_i} + q_{x_j}} \tag{17}$$

The formula (17) enables testing hypotheses about equality of differences between one unknown obtained by utilizing two different weights matrix as follows:

$$t = \frac{dx_{i,j}}{m_{dx_{i,j}}} \sim t_{f,1-\alpha} \tag{18}$$

where:

- t: student's statistics;
- $t_{f,1-\alpha}$: quantile of student's probability distribution;
- f: degrees of freedom and
- α: level of significance.

Due to efficiency of analysis in this research the hypothesis about equality of average values were tested. The test statistics in this case are as follows:

$$t = \frac{{dx_{i,j}}^{\max}}{\bar{m}_{dx_{i,j}}} \sim t_{f,1-\alpha} \tag{19}$$

$$dx_{i,j}^{\max} = \hat{x}_i^{\max} - \hat{x}_i^{\min} \quad (20)$$

$$\bar{m}_{dx_{i,j}} = \sqrt{2\bar{q}_{x_i}} \quad (21)$$

$$\bar{q}_{x_i} = \frac{1}{n}\sum_{j=1}^{n} q_{x_{i,j}} \quad (22)$$

The method explained by formulas (19-22) was possible because the variation of cofactors $q_{x_{i,j}}$ were quite small (statistically insignificant).

3 RESULTS AND DISCUSSION

Results obtained for variants of network for classical and generalized cases are given in tables as follows:

- Table 1: Variant 1 – data and null hypothesis adoption.
- Table 2: Variant 2 – data and null hypothesis adoption.
- Table 3: Variant 3 – data and null hypothesis adoption.
- Table 4: Variant 4 – data and null hypothesis adoption.
- Table 5: Variant 5 – data and null hypothesis adoption and
- Table 6: Variant 6 – data and null hypothesis adoption.

In following tables, the symbols meaning is:

- $x_{\max}$- maximal value of unknown's change
- $x_{\min}$ - minimal value of unknown's change
- $\bar{q}_x$ – average value of diagonal term of (pseudo)inverse matrix
- t – student's statistics
- Ho – null hypothesis (if "Yes" accepted, otherwise not accepted)

Table 1: Results of variant 1

	Variant 1									
	Classic Solution					Generalized solution				
	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho
R_0	-	-	-	-	-	0,12	-0,11	0,477	0,33	Yes
R_1	0,23	-0,20	0,827	0,34	Yes	0,12	-0,11	0,478	0,33	Yes
R_2	0,18	-0,21	0,827	0,30	Yes	0,10	-0,10	0,478	0,29	Yes

Table 2: Results of variant 2

	Variant 2									
	Classic solution					Generalized solution				
	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho
R_0	-	-	-	-	-	0,34	0,15	0,438	0,29	Yes
R_1	0,11	-0,14	0,713	0,21	Yes	0,31	0,19	0,436	0,18	Yes
R_2	0,11	-0,09	0,715	0,17	Yes	-0,19	-0,32	0,439	0,20	Yes
R_3	0,13	-0,14	0,715	0,23	Yes	-0,16	-0,34	0,438	0,27	Yes

Table 3: Results of variant 3

	Variant 3									
	Classic solution					Generalized solution				
	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho
R_0	-	-	-	-	-	-0,11	-0,29	0,481	0,18	Yes
R_1	0,41	0,06	0,738	0,29	Yes	-0,12	-0,28	0,480	0,17	Yes
R_2	0,49	0,18	0,823	0,24	Yes	-0,12	-0,28	0,478	0,17	Yes
R_3	0,36	0,15	0,736	0,18	Yes	-0,10	-0,29	0,480	0,19	Yes
R_4	0,55	0,35	0,691	0,17	Yes	0,80	0,80	0,403	0,01	Yes

Table 4: Results of variant 4

	Variant 4									
	Classic solution					Generalized solution				
	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho
R_0	-	-	-	-	-	-0,09	-0,25	0,513	0,16	Yes
R_1	0,45	0,13	0,747	0,26	Yes	-0,06	-0,28	0,516	0,21	Yes
R_2	0,50	0,24	0,861	0,19	Yes	-0,09	-0,25	0,516	0,17	Yes
R_3	0,52	0,22	0,860	0,23	Yes	-0,08	-0,26	0,515	0,18	Yes
R_4	0,39	0,18	0,742	0,17	Yes	-0,08	-0,27	0,514	0,19	Yes
R_5	0,55	0,38	0,678	0,15	Yes	0,84	0,83	0,375	0,01	Yes

Table 5: Results of variant 5

	Variant 5									
	Classic solution					Generalized solution				
	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho
R_0	-	-	-	-	-	0,42	0,18	0,519	0,23	Yes
R_1	0,15	-0,11	0,715	0,21	Yes	0,42	0,22	0,519	0,20	Yes
R_2	0,10	-0,09	0,715	0,16	Yes	0,16	0,01	0,519	0,15	Yes
R_3	0,15	-0,13	0,875	0,22	Yes	0,17	0,01	0,430	0,17	Yes
R_4	0,15	-0,10	0,873	0,19	Yes	-0,32	-0,49	0,520	0,16	Yes
R_5	0,12	-0,07	0,712	0,17	Yes	-0,29	-0,49	0,517	0,20	Yes

Table 6: Results of variant 6

	Variant 6									
	Classic solution					Generalized solution				
	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho	$x_{\max}$	$x_{\min}$	$\bar{q}_x$	t	Ho
R_0	-	-	-	-	-	0,04	-0,10	0,464	0,15	Yes
R_1	-0,22	-0,44	0,705	0,18	Yes	0,09	-0,12	0,606	0,19	Yes
R_2	-0,21	-0,46	0,750	0,20	Yes	-0,05	-0,22	0,532	0,17	Yes
R_3	-0,16	-0,41	0,850	0,20	Yes	-0,24	-0,40	0,605	0,14	Yes
R_4	-0,24	-0,45	0,741	0,17	Yes	-0,52	-0,66	0,462	0,15	Yes
R_5	-0,05	-0,34	0,850	0,22	Yes	-0,41	-0,60	0,605	0,17	Yes
R_6	0,01	-0,19	0,750	0,17	Yes	-0,40	-0,56	0,533	0,16	Yes
R_7	0,11	-0,16	0,706	0,22	Yes	-0,25	-0,47	0,606	0,20	Yes
R_8	0,04	-0,13	0,604	0,15	Yes	-0,27	-0,34	0,403	0,08	Yes
R_9	0,43	0,29	0,589	0,13	Yes	1,14	1,06	0,418	0,08	Yes
R_{10}	-0,04	-0,22	0,689	0,15	Yes	0,76	0,68	0,418	0,09	Yes
R_{11}	-0,06	-0,22	0,689	0,14	Yes	0,59	0,50	0,417	0,10	Yes
R_{12}	0,07	-0,05	0,589	0,11	Yes	0,45	0,34	0,418	0,12	Yes

According to the student's test statistics values obtained by (19) it is obvious that in all cases the difference between maximum and minimum values of unknowns are insignificant for the variations of weights in the interval of ±30%. This result should be considered very carefully because it was obtained with the starting assumption that initial weight matrix P was identity matrix. In this case it might be stated that, in the neighbourhood of weight matrix P when it is identity matrix, it is possible to miscalculate weights in the ±30% interval without significant loss of unknowns' accuracy.

This model of weights influence on the unknowns (in this case: benchmarks heights) is quite simplified real situations from the practice regarding the number of benchmarks and the number of measured height differences. In this research different situations that may appear in practice were not considered due to their potential complexity. The complexity of real levelling network requires analysis of each leveling network in order to find out the influence of miscalculated weights on the values of unknows.

This paper represents the results of theoretical research and proposed the method for determination of miscalculated weights influence on the unknowns. This method also avoids the complex mathematical models for inverse matrix, which is sum of two matrices, calculation. Applying this method on the real levelling network could check the sensitivity of unknows on the weights' miscalculation.

Further investigation shall be carried out by encompassing different levelling networks from practice.

4 CONCLUSIONS

This research was conducted with the aim of determining whether miscalculation of weights influences the values of adjusted unknowns. The simulation was provided for six different levelling networks starting from simplest and gradually increasing their complexity. The research was provided by utilizing the classical LSM and the generalized method. The weight matrix started as an identity matrix with its values varied in the interval of ±30% utilizing random function. The results obtained showed that in every case there were no reasons to reject null hypothesis i.e. the differences obtained by 100 variations of weight matrix resulted with the same values of unknown in a statistical sense. This conclusion was obtained both for classical and generalized solutions. This result should be considered very carefully because it represents simplified levelling networks obtained by simulation. For further research it is necessary to provide proposed method on the real levelling network with results obtained from in-field measurements.

REFERENCES

[1] Berber, M. (2006). Robustness Analysis of Geodetic Networks. Ph.D. dissertation, Department of Geodesy and Geomatics Engineering, Technical Report No. 242, University of New Brunswick, Fredericton, New Brunswick, Canada, 121 pp.

[2] Delčev, S., Gučević, J., Miljković, S., Ogrizović, V. (2019): THE METHODS OF CALCULATING WEIGHTS OF HEIGHT DIFFERENCES, Contemporary Achievements in Civil Engineering 23-24. April 2019. Subotica, SERBIA

[3] Allen, F. J., Fishkind, D. E. (1998): "The Moore--Penrose generalized inverse for sums of matrices." SIAM Journal on Matrix Analysis and Applications 21.2, pp 629-635.

[4] Mihailović, K., Aleksić, R.I. (1994): Deformaciona analiza geodetskih mreža, "GEOKARTA", Beograd.

[5] Mueller, K. S. (1981): "On the inverse of the sum of matrices." Mathematics magazine 54.2, pp 67-72.

[6] Niemeier, W., Tengen, D. (2017). Uncertainty assessment in geodetic network adjustment by combining GUM and Monte-Carlo-simulations. Journal of Applied Geodesy, 11(2), 67-76.

[7] Perović, G. (1989): Račun izravnanja, Knjiga 1, Teorija grešaka merenja, Naučna knjiga, Građevinski fakultet, Beograd.

[8] Perović, G. (2015): Teorija grešaka merenja, AGM KNJIGA, Beograd, ISBN 978-86-86363-55-8.