DUALITY OF BUCKLING AND VIBRATION PHENOMENA USING FINITE STRIP METHOD

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Summary: *It is well recognized that when using exact theory, as opposed to the more usual approximate finite element method, there exists a duality between buckling and natural vibration of an elastic structure, even though the former is a static problem whereas the latter is a dynamic one. The solution procedures for the both problems are analogous because both can be essentially represented by a transcendental eigenvalue problem, the eigenvalues being natural frequencies in vibration problems and load factors in buckling problems. Thus the elastic buckling of a structure is often regarded as the degenerated case of its natural vibration at zero frequency. In this paper duality is proved for the both elastic and viscoelastic (or damage) structure using semi-analytical finite strip method (FSM) and rheological-dynamical analogy (RDA). The governing dynamic RDA modulus has been derived in [1]. This paper presents an investigation of composite thin-walled wide-flange columns. Numerical examples showing the theoretical considerations are presented and agree with the experimental data. Two reference Open Source Software implementations are provided: for approximating the natural frequency from stress via physical duality (and vice versa) and for the visual analysis of the physical duality approximation effectiveness.*

Keywords: *Duality, FSM, RDA, dynamic RDA modulus, Composite thin-walled wide-flange column, numerical analysis, accuracy of numerical evaluation*

1. INTRODUCTION

The conventional FSM is based on the eigen-functions for the vibrating beam, and proved to be efficient tool for analyzing a great deal of structures for which both geometry and material properties can be considered as constants along a main direction,

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straight or curved, while only the loading distribution may vary. As far as linear analysis is concerned, it takes advantage of the orthogonality properties of eigen-functions in the stiffness matrices formulation.

The analysis of a variety of elastic structures subjected to static loads lead to the same matrix equation of

$$\mathbf{Kq} = \mathbf{Q} \tag{1}$$

in which **K** is the stiffness matrix of structure, **q** the vector containing all nodal displacement parameters, and **Q** the vector containing all nodal forces.

If the structure is moving then it is also possible to reduce the dynamic problem to a static one by applying D'Alembert's principle of dynamic equilibrium in which an inertial force equal to the product of the masss and the acceleration is assumed to act on the structure in the direction of negative acceleration. Thus at any instant ot time the equibrium equation for a structure in which both damping and external excitation forces are assumed to be non-existent is

$$\mathbf{Kq}(t) = -(\mathbf{M}+\mathbf{m})\ddot{\mathbf{q}}(t) \tag{2}$$

where $\mathbf{q}(t)$ is now a function of time and .. reprents $\partial^2/\partial t^2$. In the above equation **M** is a diagonal matrix of concentrated or line masses at the nodal lines, and is simply equal to zero when no such concentrated or line masses are acting on the structure, and **m** is an overall mass matrix of the structure assembled from individual strip consistent mass matrices $\mathbf{m}^s$. The assembly process for mass matrices and for stiffness matrices are identical [2].

For free vibration, the system is vibrating in a normal mode, and it is possible to make the substitutions

$$\mathbf{q}(t) = \mathbf{q}\, sin\, \omega t \,, \qquad \ddot{\mathbf{q}}(t) = -\omega^2 \mathbf{q}\, sin\, \omega t \tag{3}$$

into (2) to obtain

$$\left(\mathbf{K} - \omega^2 \mathbf{m}\right)\mathbf{q} = \mathbf{0} \tag{4}$$

where ω is the natural frequencies of the modes and the common term $sin\,\omega t$ has been concelled out.

It is possible to transform (4) into

$$\left(\mathbf{K}^{-1}\mathbf{m}\right)\mathbf{q} = \frac{1}{\omega^2}\mathbf{q} \tag{5}$$

which becomes, therefore, a transcendental eigenvalue problem with the form

$$\mathbf{Ax} = \lambda \mathbf{x} \,. \tag{6}$$

Comparatively little work has been done on the application of the finite strip to stability problems, although vibration and stability share many similar features and both require the determination of eigenvalues and eigenvectors.

Two types of column failure (buckling) are well known for wide-flange sections: local buckling and overal (Euler) column buckling. In local modes, only non-linear terms such as square derivatives of transverse displacement *w* need to be included (von Karman approach).

It is obvious that if the total potential energy of the strip, i.e. the sum of strain energy due to bending, potential energy due to nodal line forces, and the additional potential energy due to the initial stress, is now minimized with respect to the nodal displacement parameters, the following relationship would be obtained:

$$\mathbf{Sq} + \mathbf{S}_G\mathbf{q} = \mathbf{Q} \tag{7}$$

in which $\mathbf{S}_G$ is referred to as the geometric stiffness matrix of strip or initial stress matrix and takes up the same sign as the stresses.

Upon assembly of the contributions from all the strips an overall set of equilibrium equations is established,

$$\mathbf{Kq} + \mathbf{K}_G\mathbf{q} = \mathbf{Q} \tag{8}$$

For linear stability, the nodal forces are zero and it is therefore possible to arrive at eigenvalue equations similar to the ones given in (4),

$$\left(\mathbf{K} + \lambda\mathbf{K}_G\right)\mathbf{q} = \mathbf{0} \tag{9}$$

with λ being scaling factor related to the critical load

$$\sigma = \sigma_{cr} = \frac{\lambda_{min}}{2 \cdot t} \tag{10}$$

where t is the web (flange) thickness.

2. DUALITY OF BUCKLING AND VIBRATION

The propagation of mechanical waves (or stress waves) with transition from the short-time modulus of elasticity (E_D) to the long-time one (E_H) represents a physical basis for the analogy between two different physical phenomena, the rheological and the dynamical. Generally speaking, the RDA is derived in order to solve dynamic problems, but it can be used in the analysis of quasi-static loading considering the corresponding limit values of the derived analytical expressions. The governing dynamic RDA modulus has been derived in [1].

$$E_R = \frac{1+\varphi+\delta^2}{\left(1+\varphi\right)^2+\delta^2}E_H\,, \qquad \underset{lim\delta\to 0}{E_R} = E_R = \frac{E_H}{1+\varphi}, \qquad \delta = \frac{\omega_\sigma}{\omega}\,. \tag{11}$$

φ is the structural-material creep coefficient and ω_σ is the frequency of excitation.

Elastic buckling analysis by the FSM was pioneered by Yoshida [3] and Przemieniecki [4]. Plank and Wittrick [5] generalized the approach to include shear loading, by using complex harmonic functions. They pointed out that duality exists between the free-vibration behavior and buckling under unifrom end compression.

$$\omega_{im} = \frac{\sigma_{im}}{\rho}\left(\frac{m \cdot \pi}{a}\right)^2 \tag{12}$$

where i = number of degrees of freedom (DOF), ρ = mass density, m = number of series term and a = length of the column. In this paper duality is proved for the both elastic and viscoelastic structure using the FSM and RDA.

3. SOFTWARE IMPLEMENTATION

The fsm_eigenvalue project [6] provides a reference Open Source Software implementation for parametric modeling of buckling and free vibration in prismatic

structures, performed by solving the eigenvalue problem in the harmonic coupled finite strip method (HCFSM).
The fsm_eigenvalue project takes the semi-analytical finite strip model data file (geometry, materials, loading) as its input and then performs its computations as a parameter sweep over 4 separate dimensions:

- D1: Performs the buckling and free vibration analysis
- D2: Iterates over all strip lengths, in the range specified by the input
- D3: Iterates over all strip thicknesses, in the range specified by the input
- D4: Iterates over all modes, in the range specified by the input.

The HDF5 (Hierarchical Data Format 5) [7] format has been selected for storing the computed results, because of its innate ability to efficiently store and organize large amounts of numerical data. HDF5 is an extensible and open standard, comprised of platform independent technologies which are available under Open Source licenses. HDF5 format has been designed with the objective of creating data storages that are self-descriptive, flexible, and have extremely fast and efficient access patterns to the stored data.

For each combination of mode, strip length and thickness the fsm_eigenvalue project outputs a large amount of numerical data: critical stress, natural frequency, their approximations via physical duality, approximation errors, minimal critical stress vector, minimal natural frequencies vector, etc. To approximate the natural frequency from stress via physical duality (and vice versa) a separate Open Source programming library has been created, named physical_dualism [8].

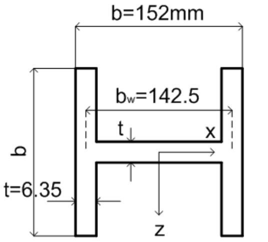
When larger finite strip models are analyzed, such as the example data files provided by the fsm_eigenvalue project – that have 51 modes, 7800 strip lengths (100-4000 mm with a 0.5 mm iteration step) and 140 strip thicknesses (2-9 mm with a 0.05 mm iteration step), computed results can amount to over 50 GB of compressed data per HDF5 file.

Such data can’t be inspected manually, so it was necessary to build a new Open Source project for automated visualization and modal analysis of the parametric model of buckling and free vibration in prismatic shell structures, named fsm_modal_analysis [9]. This software has been used to generated Figs. 1, 2, 3 and 4 within this paper.

4. APPLICATIONS

The simply supported ideally straight thin-walled wide-flange H-section column of length a = 2310 mm consists of a web and two flanges of side b = 152 mm [10]. The thickness t is 6.35 mm. The column is compressed axially. The bending stiffness EI=89.6 kNm^2 about the weak axis of the analysed column was computed from the information provided by the manufacturer for each section following the methodology developed by Barbero and Tomblin [11]. This information includes the type of fibres and matrix material, the local orientation of the fibres and the fibre content in the cross-section. Tab. 2 shows the flange and web bending stiffness components for 152 mm x 152 mm x 6.35 mm pultruded WF H-section column.

Table 1. Weak axis bending stiffness components of flanges and web: test data [10]

	Stiffness component (kNm)	Flanges	Web
	D_{11} D_{22} D_{12} D_{66}	302.95 116.28 44.35 38.52	0.46217 0.21052 0.08267 0.06736

The buckling behaviour of ideally straight column to axial loading and small transverse concentrated load has already been investigated by the HCFSM [12]. Milašinović [2] developed a harmonic coupled FSM by including the Green-Lagrange strain terms in the formulations, thus bending and membrane are coupled in the geometric stiffness to give more accurate buckling behaviour prediction, especially in large-deflection and elastic post-buckling analysis. The following approximate functions are used for the simply flat shell strip.

$$
\begin{aligned}
u_0 &= \mathbf{A}_u^u \mathbf{q}_u^u = \sum_{m=1}^{r} Y_{um}^u \mathbf{N}_u^u \mathbf{q}_{um}^u = \sum_{m=1}^{r} Y_{um}^u \left[1-x/b \quad x/b\right]\mathbf{q}_{um}^u ,\\
v_0 &= \mathbf{A}_u^v \mathbf{q}_u^v = \sum_{m=1}^{r} \frac{a}{m\pi} Y_{um}^v \mathbf{N}_u^v \mathbf{q}_{um}^v = \sum_{m=1}^{r} \frac{a}{m\pi} Y_{um}^v \left[1-x/b \quad x/b\right]\mathbf{q}_{um}^v ,\\
w &= \mathbf{A}_w \mathbf{q}_w = \sum_{m=1}^{r} Y_{wm} \mathbf{N}_w \mathbf{q}_{wm} = \sum_{m=1}^{r} Y_{wm} \left[N_1 \quad N_2 \quad N_3 \quad N_4\right]\mathbf{q}_{wm} ,\\
N_1(x) &= 1-3(x/b)^2+2(x/b)^3, \qquad N_2(x)=b\left[x/b-2(x/b)^2+(x/b)^3\right],\\
N_3(x) &= 3(x/b)^2-2(x/b)^3, \qquad N_4(x)=b\left[-(x/b)^2+(x/b)^3\right],\\
& Y_{um}^u(y) = sin(m\pi y/a) = Y_{wm}(y),\\
& Y_{um}^v(y) = dY_{um}^u/dy = (m\pi/a)\cos(m\pi y/a), \qquad m=1,2,3,...
\end{aligned}
\tag{13}
$$

The material properties in Table 3 for the implementation of HCFSM are obtained as detailed in [12], and there are used in this paper.

Table 2. Material properties of flange and web

Material Properties	Elastic		Viscoelastic	
	Flanges	Web	Flanges	Web
E_x (N/mm^2)	62786.25	52906.25	20928.75	17635.42
E_y	24098.98	24098.98	8032.99	8032.99
μ_x	0.38	0.39	0.38	0.39
μ_y	0.15	0.18	0.15	0.18
G	1805287.39	3156.91	1805287.39	3156.91

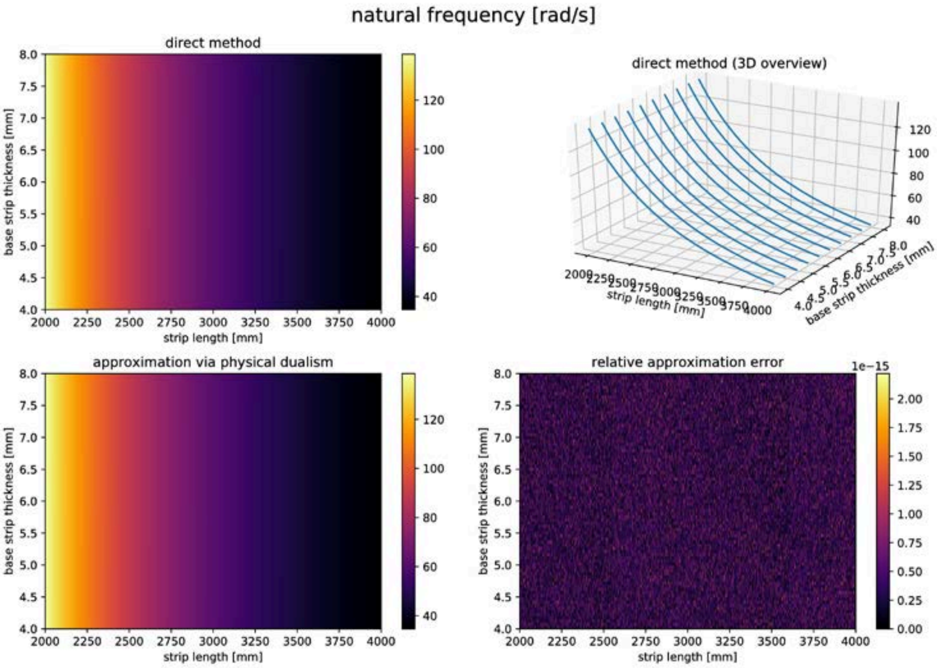


Figure 1. Elastic natural frequencies

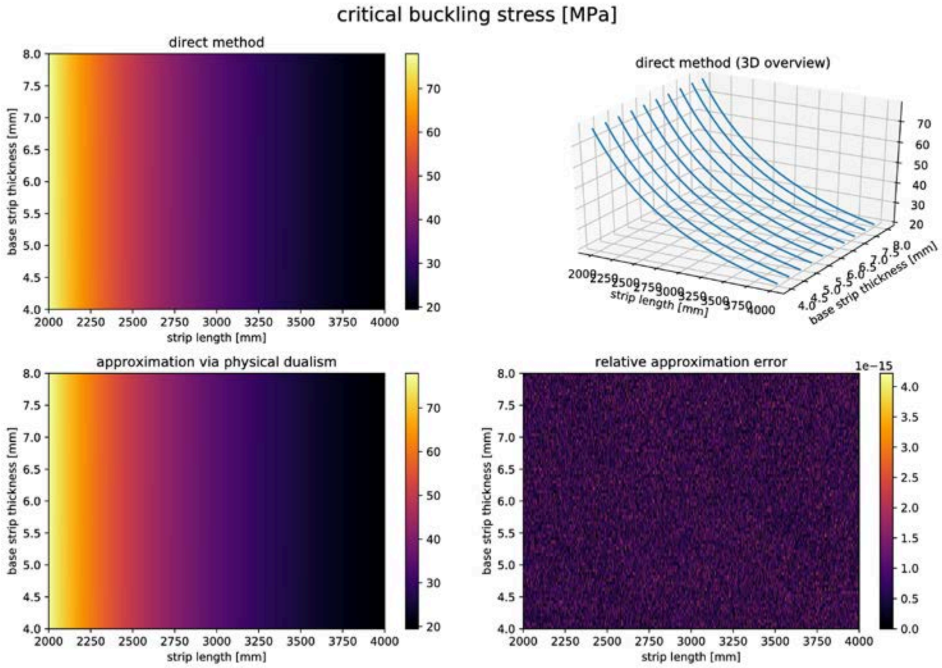


Figure 2. Elastic critical buckling stresses

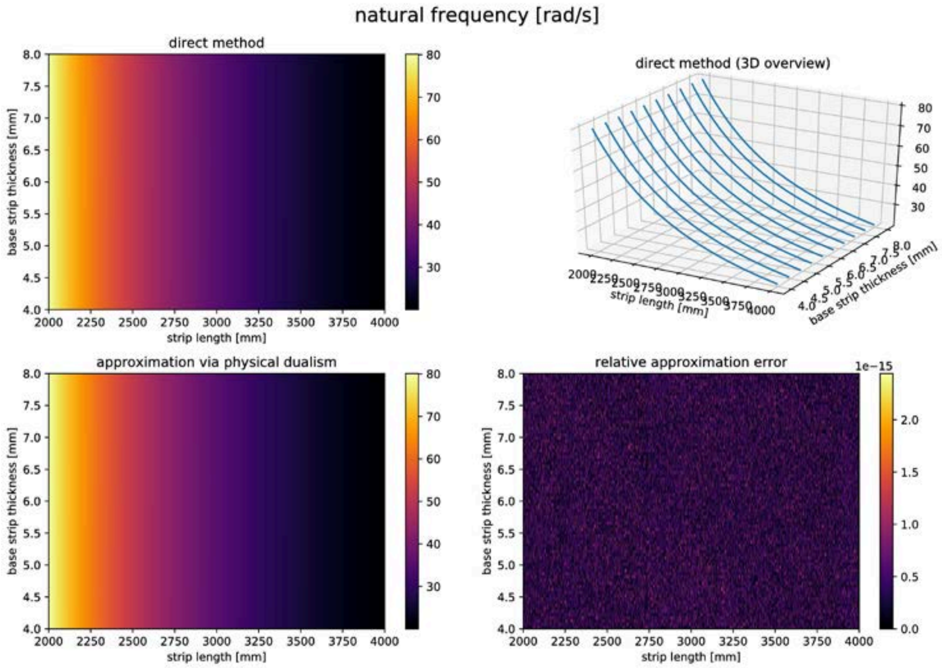


Figure 3. Viscoelastic natural frequencies

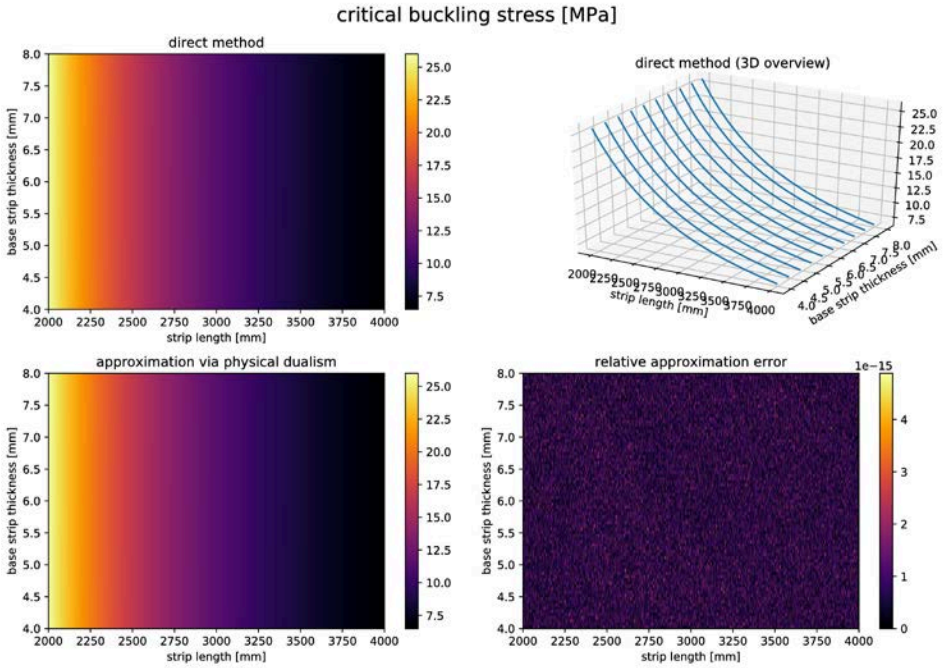


Figure 4. Viscoelastic critical buckling stresses

Fig. 1 shows the results of natural frequencies for the elastic solutions of the problem. The column length has been successively increased from 2000 to 4000 mm, with a step of 0.5 mm. There is a noticeable lag between the viscoelastic, Fig. 3 and the elastic natural frequencies.

Fig. 2 presents the elastic buckling curve for all lengths of columns from 2002 to 4000 mm, with a step of 0.5 mm. Viscoelastic buckling stress, Fig. 4 lags behind the elastic buckling stress across all modes, which is a consequence of the visoelastic behaviour of materials. The viscoelastic behaviour is characterized by the delay time T^D [1]. As the length of the column is large, the observed lag increases. At the column length of 2310 mm, the computed elastic critical stress of the first global mode correspond to test results [10]. Also, the computed viscoelastic stress at 2310 mm of the first global mode has been previously confirmed in [12].

5. CONCLUSIONS

Numerical examples were computed by application of extensive hybrid parallelization. Results from the numerical studies for all lengths of columns from 2000 to 4000 mm, show that columns have all elastic and inelastic characteristics in the same (first) mode. Because of that the mode interaction is not occurred. Numerical examples showing the theoretical considerations are presented and agree with the experimental data at the column length of 2310 mm. Two reference Open Source Software implementations are provided: for approximating the natural frequency from stress via physical duality (and vice versa). In this paper duality is proved for the both elastic and first time for viscoelastic columns using the RDA.

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ДУАЛИТЕТ ПОЈАВА ИЗВИЈАЊА И ВИБРАЦИЈА КОРИШТЕЊЕМ МЕТОДА КОНАЧНИХ ТРАКА

Резиме: *Добро је познато, када се користи аналитичка теорија за разлику од чешће коришteног апроксимативног метода коначних елемената, да постоји дуалитет између извијања и сопствених вибрација еластичне конструкције, иако је први статички проблем, док је други динамички. Процедуре решавања за оба проблема су аналогне јер се оба могу представити трансцендентални* *проблемом сопствених вредности, сопствене вредности су природне фреквенције у проблемима вибрација, а фактори оптерећења у проблемима извијања. Због овога се еластично извијање конструкције често посматра као дегенерисани случај свог природног вибрационог стања са нутом сопственом фреквенцијом. У овом раду дуалитет је доказан за обе, еластичну и вискоеластичну (оштећену) конструкцију коришћењем пулуаналитичког метода коначних трака (МКТ) и реолошко динамичке аналогије (РДА). Владајући динамички модул је изведен у [1]. Овај рад представља истраживање композитних танкозидних стубова са широким фланшама. Нумерички примери приказују теоријска разматрања, а у сагласности су са експерименталним подацима. Два референтна 'Open Source Software' су обезбеђена за прорачун сопствених фреквенција преко критичних напона према физичком дуалитету (и обрнуто) као и за визуелну анализу ефикасности апроксимације физичким дуалитетом.*

Кључне речи: *Дуалитет, МКТ, РДА, динамичи РДА модул, Композитни танкозидни стуб са широким фланшама, нумеричка анализа, тачност нумеричке евалуације*