A NOTE ON THE EQUILIBRIUM ANALYSIS OF INCLINED PLATE-BANDE

Dimitrije Nikolić [1]
UDK: 624,04
DOI:10.14415/konferencijaGFS2017.040

Summary: *In 1807, Thomas Young published an article dealing with different structural forms of false arches. One of considered problems was the stability of two inclined plate-bandes leaning against each other. In the present paper, after a geometric formulation within equilibrium (static) approach, the problem is analytically treated, and the closed-form expression of thrust line is derived. In addition, limit equilibrium state is analysed, and minimum thickness to length ratio as the function of the inclination angle is determined. Accordingly, Young's solution turns out to be incorrect, since instead of 15, even 20 bricks are stable within the plate-bande inclined at 60 degrees.*

Keywords: *thrust line, inclined plate-bande, limit equilibrium analysis, Thomas Young*

1. INTRODUCTION

In 1807, Thomas Young, under a pseudonym [1], published an article [2] dealing with different structural forms of false arches. One of considered problems was related to two inclined brick plate-bandes (straight arches) leaning against each other, and the maximum number of bricks that are thereby stable. After derived geometrical construction, which appears to be incorrect [1], Young concluded that "if the aperture be equilateral, 15 common bricks on each side will stand, but 16 will give way at the sixth joint from the summit" [2, p. 247], as shown in Fig. 1a. Barlow [3] analysed the same problem, but provided only the numerical result for the inclination of 45°.
He concluded that "thickness must be 0,1464 of the length" and that critical joint appears at the distance being equal to "0,3535 of the length, from the upper extremity"; furthermore, he noticed that this thickness is sufficient for any other inclination [3, pp. 165–166]. Recently, Huerta discussed Young's result and concluded that "the difference between Young's solution and the correct solution is small", and that "exact calculation, resolving transcendental equation to obtain the limit thickness, gives very nearly a proportion of 1/6 and the position of the joint of rupture at circa 1/4 of the length from the top" [1, p. 417].
In 1730, Couplet [4] tried to determine the minimum thickness of a semicircular arch of uniform thickness under self-weight, which is today known as Couplet's problem. In 1904, on the base of thrust line i.e. limit equilibrium analysis, Milankovitch [5] provided the correct solution, and recently, the same approach was applied to the elliptical [6] and pointed arches [7]. Albeit the geometry of inclined straight arch (or two inclined plate-

[1] Dimitrije Nikolić, M.Arch., University of Novi Sad, Faculty of Technical Sciences, Department of Architecture, Trg Dositeja Obradovića 6, Novi Sad, Serbia, e-mail: dima@uns.ac.rs

bandes leaning against each other) is very simple, precise mathematical elaboration of this apparently simple problem is not provided. Thus, the aim of this paper is to apply geometrical formulation (static approach) of limit equilibrium analysis, i.e. derivation of limit thrust line, and thus once again to reassess the previous results.

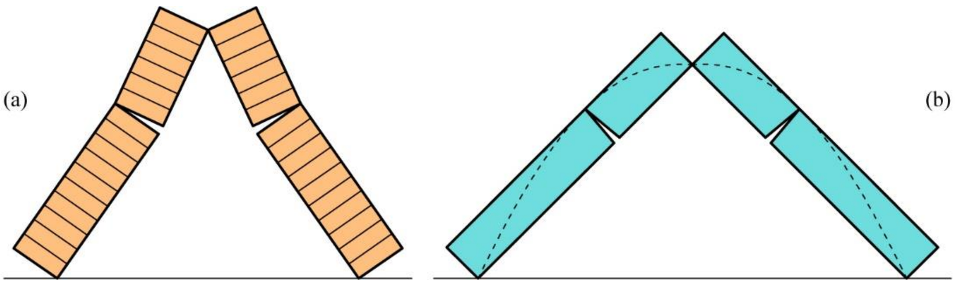

Figure 1. Two inclined plate-bandes leaning against each other:
(a) Young's solution for α=60°, (b) Barlow's solution for α=45°

2. ANALYTICAL MODELLING

Analytical modelling of the problem is based on thrust line analysis, where thrust line represents the locus of the application points of the resultant thrust forces at the joints between the voussoirs of a structure.
Therewith, common assumptions about masonry properties are adopted: no tension strength, infinite compression strength and sliding cannot occur. Thus, the possibility of failure due to material strength or due to sliding is eliminated, permitting only the collapse due to instability, by relative rotation of structure's parts around the edge of the joint of rupture. Accordingly, if a structure is of sufficient thickness, and a thrust line is lying everywhere within the structure's boundary (between intrados and extrados), the structure is safe [8].
Consider now two plate-bandes of the length *l*, thickness *t* and half-span *s*, leaning against each other at the angle α, as shown in Fig. 2d. Their common point *B* and two points of support define unique thrust line.
Due to the symmetry, in the following analysis only half-arch is considered. The horizontal thrust *H* is the only one that can be transmitted at the crown; on the other side, there is inclined reaction force *R* acting at the point of support *S*, as shown in Fig. 2a. Weight *W*, represented by the area of plate-bande, is acting at the corresponding centre of gravity. The force polygon shown in Fig. 2b graphically expresses the equilibrium of the system. The origin of the Cartesian coordinate system is set at the point *B*, and the abscissa is set along intrados, as shown in Fig. 2a,c. Since inclined plate-bande can break along the direction of joint between bricks which is perpendicular to the intrados and extrados, normal stereotomy is employed.
Hence, the distance *x* is measured along the abscissa and defines the generic section which is parallel to the ordinate.

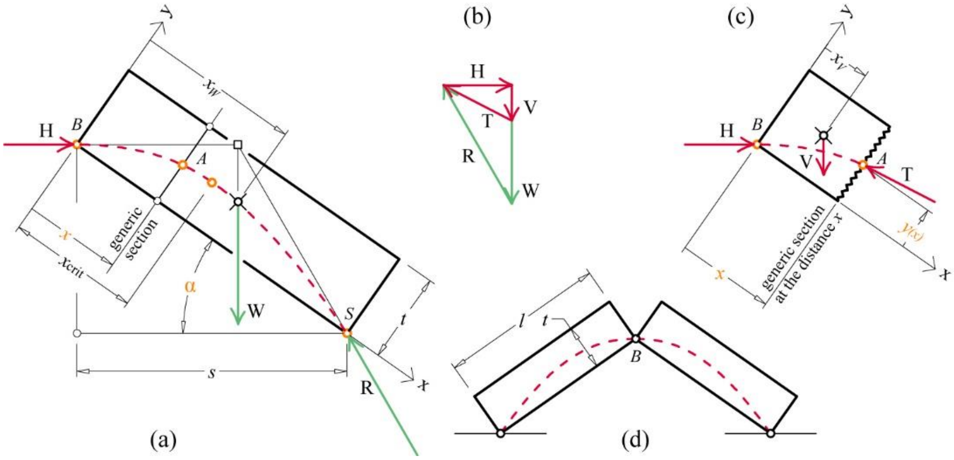

Figure 2. Inclined plate-bande: (a) geometric parameters, (b) force polygon, (c) free-body diagram of the isolated finite top portion up to the generic section, (d) two inclined plate-bandes leaning against each other

Rotational equilibrium about the point *S* is expressed by the following equality:

$$H \sin \alpha\, l + \mathrm{W} \sin \alpha\, l / 2 = \mathrm{W} \cos \alpha\, l / 2 \tag{1}$$

whereas $W = l\ t$ is the weight of the inclined plate-bande, and $l = s \sec\alpha$. Thus, one can derive the value of horizontal thrust:

$$H = \frac{t(s - t \sin \alpha)}{2 \sin \alpha} \tag{2}$$

One can conclude that *H* equals zero for $t = s \csc\alpha$, when the vertical line of action of the weight *W* passes through the point *S*. Consider now the finite portion up to the generic section at the distance *x*, shown in Fig. 2c. Its weight $V = x\ t$ is represented by the corresponding area and is applied at its centroid. The resultant thrust force *T* at a generic section at the distance *x* together with its point of application *A* is uniquely determined from the force and moment equilibrium of the finite portion of the arch; it can be done either graphically with the force polygon (Fig. 2b,c), or analytically by solving equilibrium equations. Accordingly, from rotational equilibrium about point *A* follows:

$$H\ x \sin \alpha - H\ y \sin \alpha = V \sin \alpha \left(y - \frac{t}{2} \right) + \frac{1}{2} V\ x \cos \alpha \tag{3}$$

whereas the horizontal thrust *H*, given by Eq. (2), and the weight *V* are decomposed to the normal (abscissa) and shear (ordinate) direction. Finally, from the previous equality, one can determine the distance *y* between the thrust line and the intrados, deriving the closed-form expression for the thrust line within inclined plate-bande:

$$y(x)=\frac{x^2\cos\alpha - sx}{t\cos\alpha - 2x\sin\alpha - s\cot\alpha} \tag{4}$$

Critical point, referring to joint (generic section) where the thrust line approaches closest to the extrados, is determined according to the maximum of Eq. (4) (for the brace of the hyperbola which corresponds to the existing, real, part of the thrust line; see Fig. 5b); thus, derivative of Eq. (4) is:

$$y'(x)=\frac{\cot\alpha(s-t\sin\alpha)(s-2x\cos\alpha)-2x^2\sin\alpha\cos\alpha}{(2x\sin\alpha+s\cot\alpha-t\cos\alpha)^2} \tag{5}$$

and from $y'(x) = 0$, the positive value x, being the abscissa of the critical point, is solved:

$$x_{\mathrm{crit}}=\frac{1}{2}\left[\cot\alpha(t-s\csc\alpha)+\csc a\sqrt{\sin\alpha(s\csc\alpha-t)\ \left(s\cot^2\alpha+2s-t\cos\alpha\cot\alpha\right)}\right] \tag{6}$$

Substitution of Eq. (6) into Eq. (4) gives the ordinate of the critical point, representing the greatest distance between the intrados and the thrust line:

$$y_{\mathrm{crit}}=\frac{1}{4}\left\{2s\csc^3\alpha-2\cot\alpha\csc\alpha\ \left[t\cos\alpha+\sqrt{\sin\alpha(s\csc\alpha-t)\ \left(s\cot^2\alpha+2s-t\cos\alpha\cot\alpha\right)}\right]\right\} \tag{7}$$

Equalization of Eq. (7) with the thickness t, resolving for t and the division by s gives the minimum thickness to span ratio as the function of the inclination angle:

$$t/s_{\min}(\alpha)=\frac{1}{2}\tan\left(\frac{\alpha}{2}\right) \tag{8}$$

Furthermore, the ratio between the thickness t and the length l of the inclined plate-bande is:

$$t/l_{\min}(\alpha)=\frac{1}{2}\tan\left(\frac{\alpha}{2}\right)\cos\alpha \tag{9}$$

The inclined plate-bande of this proportion is of minimum thickness, i.e. it is on the point of collapse (limit equilibrium state), and can accommodate only one admissible (limit) thrust line (see Fig. 5d). One can see that this value depends only on the value of the inclination angle α. Furthermore, when the minimum thickness is assumed, the abscissa x_{crit} of the critical point, given by Eq. (6), is simplified to:

$$x_{\mathrm{crit}}(t_{\min})=\frac{1}{2}s=\frac{1}{2}l\cos\alpha \tag{10}$$

This enables simple determination of the position of the joint of rupture.

3. RESULTS AND DISCUSSION

Equations derived in the previous section refer to inclined plate-bandes made of bricks of infinitesimal thickness. Numerical values for theoretical minimum thickness for the various values of the inclination angle α are calculated according to Eqs. (8) and (9) and are given in Table 1. In addition, the correlation between minimum thickness to span ratio or (*t/s*) minimum thickness to length ratio (*t/l*) and the inclination angle α is traced in the graph presented in Fig. 3a.

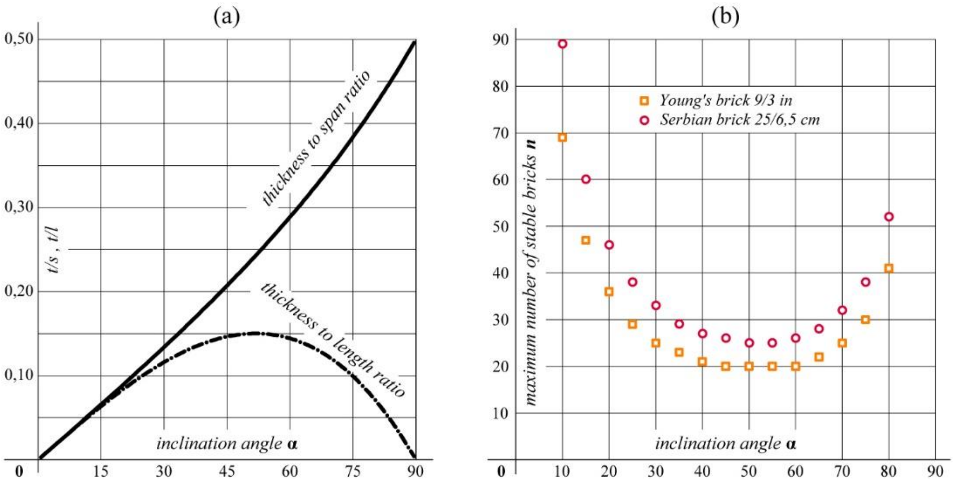

Figure 3. (a) Graph showing minimum thickness to span t/s *and thickness to length* t/l *ratio for various inclination angles, (b) maximum number of bricks within plate-bande*

On the other hand, when the bricks of finite thickness are considered, the maximum number of bricks, laid one above other within the plate-bande that will be stable, depends on their proportions. Thickness to length ratio (*t/l*) of a plate-bande can be expressed as follows:

$$\frac{t}{l} = \frac{t}{n\,a} = \frac{\lambda}{n} \tag{11}$$

whereas *n* is the number of bricks of thickness *a*, and λ represents the length to thickness ratio (*t/a*) of the brick. Since *t/l* must be greater than $t/l_{\min}$, given by Eq. (9), the maximum number of bricks of proportion λ, is given by:

$$n \le \frac{\lambda}{t/l_{\min}(\alpha)} \tag{12}$$

In his article, Young does not explicitly specify the dimensions of the "common brick" that he used in his analysis (in further text: Young's brick). However, relating to his Fig. 18 [2, plate VIII], redrawn here in Fig. 3a, the following is stated: "…an equilateral aperture, constructed of 8 common bricks on each side, and without cement of any kind, CD will be 9.3 inches, DE 2.7, and FG 21…" [2, p. 247]. Accordingly, one can conclude that brick's thickness equals 3 (21 divided by 7), and the following equalities can be set:

$$21\cos 60 - t\cos 30 = 2{,}7 \tag{13a}$$

$$t\cos 30 + 3\cos 60 = 9{,}3 \tag{13b}$$

From any of the equations, one can easily deduce the length t of the Young's bricks (that is the thickness of the inclined plate-bande), which equals 9. Hence, the thickness to length ratio of Young's brick is 3/9, rather than 3/8,75 as incorrectly deduced by Huerta [1, p. 417] (however, this difference (8,75 or 9) does not affect the final result).

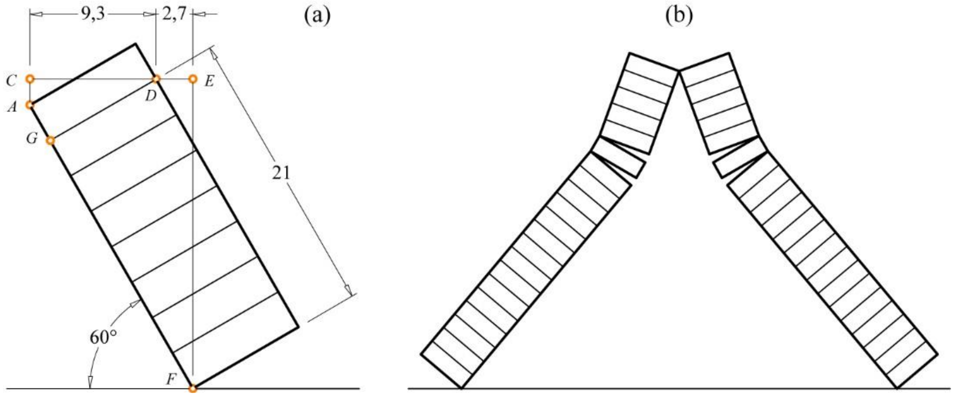

Figure 4. (a) Proportions of Young's bricks (*redrawn from* [2]), *(b) collapse of two inclined plate-bande leaning against each other, each made of 21 bricks*

As mentioned in the Introduction, Young stated that, in the case when α=60°, 16 bricks will collapse at the sixth joint from the top. That implies thickness to length ratio of inclined plate-bande between 9/45 and 9/48, i.e. 0,2 < t/l < 0,1875, and the rupture occurs at 6/16 of the length from the top. However, the correct solution for the minimum thickness, according to Eq. (9), is t/l_{min}≈0,14434. Therefore, with respect to Eq. (12), one can conclude that even 20 bricks of such proportion would still stand (t/l=3/20=0,15), and that 21 bricks would collapse (t/l=3/21=0,1429). In addition, Huerta's comment that "exact calculation…gives very nearly the proportion of 1/6", which is 3/18≈0,1667, turns out to be incorrect (although Huerta's Fig. 15 [1, p. 417] appears very close to the correct). On the other hand, with respect to Eq. (10), when the minimum thickness is assumed, critical joint is at 1/4 of the length from the top. However, when the bricks of given proportions are considered, Huerta's note that "the position of the joint of rupture [is at] circa 1/4 of the length from the top" is correct, since plate-bande breaks around the fifth brick, i.e. the thrust line cuts the extrados at the fifth and the sixth joint from the top, as shown in Fig. 4b.

As regards the inclination angle α=45°, the thickness to length ratio (t/l=0,14645) and the position of the joint of rupture (x_{crit}=0,35355 l), given by Eqs. (9) and (10), respectively, are in accordance with the Barlow's solution. However, his remark that "whether the inclination was greater or less than 45°, the curve [thrust line] fell within the thickness" turns out to be incorrect (see Table 1 and the graph shown in Fig. 3a). Namely, the derivative of Eq. (9) is:

$$t/l_{\min}'(\alpha)=\frac{1}{2}\left(\cos\alpha-\frac{1}{1+\cos\alpha}\right) \tag{14}$$

Equalizing Eq. (14) with zero, gives the maximum of Eq. (9), representing the value of the inclination angle α which requires theoretically the greatest value of minimum thickness to length (*t/l*) ratio:

$$\alpha_{\max}(t/l_{\min})=\arccos\left(\frac{\sqrt{5}-1}{2}\right)\approx 51{,}8° \tag{15}$$

Maximum number of Young's bricks within inclined plate-bande that is stable, for various values of inclination angle α, is provided in Table 1. Therewith, the values regarding bricks commonly used in Serbia, with length to thickness ratio 25/6,5, are given as well. This is also graphically presented in Fig. 3b.
On the base of Eqs. (9) and (10), one can derive very simple geometrical construction, presented in Fig. 5a, for the detection of minimum thickness for a given intrados. The procedure is as follows: (I) bisect the given inclination angle (straight line AD); (II) the half of the obtained straight line segment CD represents the minimum thickness value.

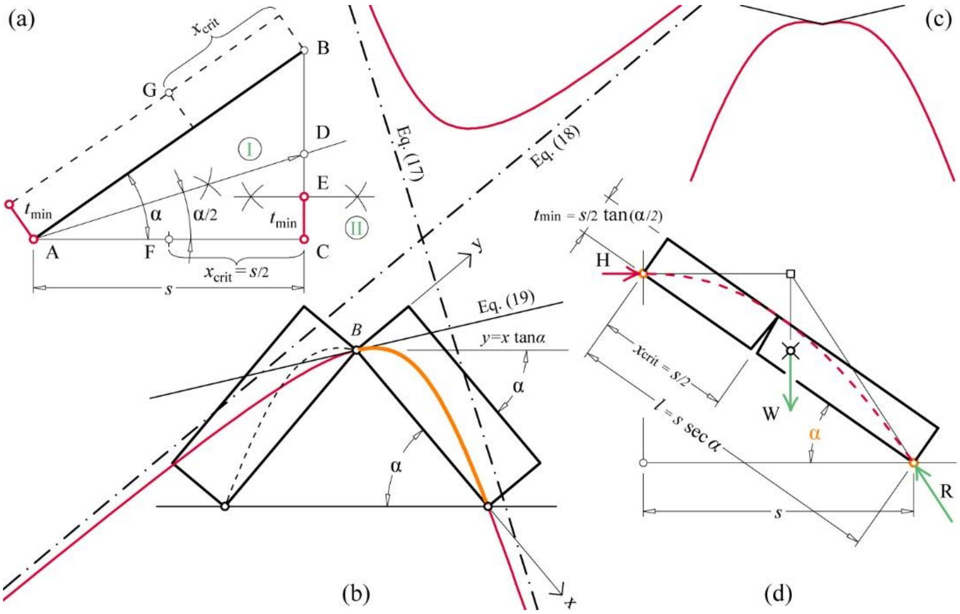

Figure 5. (a) Geometrical construction for the determination of minimum thickness and the position of critical joint, (b) hyperbolic thrust line (c), cusp of the thrust line at the crown, (d) inclined plate-bande of minimum thickness

Table 1. Minimum thickness to span (t/s) and thickness to length (t/l) ratio, as well as the maximum number of bricks within inclined plate-bande for various inclination angles

α [°]	15	22,5	30	35	40	45	50	55	60	67,5	75
t/s	0,0658	0,0995	0,1340	0,1576	0,1820	0,2071	0,2332	0,2603	0,2888	0,3341	0,3837
t/l	0,0636	0,0919	0,1160	0,1291	0,1394	<u>0,1464</u>	0,1499	0,1493	0,1443	0,1279	0,0993
Young's bricks	47	32	25	23	21	20	20	20	<u>20</u>	23	30
Serbian bricks	60	41	33	29	27	26	25	25	26	30	38

At the end, several properties of thrust line regarding its shape are considered. The expression of the thrust line, given in Eq. (4), is rational function which represents hyperbola (see Fig. 5b). Since the degree of the numerator is one degree greater than the denominator, the graph has oblique asymptote; the division of these two polynomials, ignoring the remainder, gives its expression:

$$y(x) = -\frac{1}{2} x \cot\alpha + \frac{1}{4\sin\alpha}\left[s\left(2+\cot^2\alpha\right) - t\cos\alpha\cot\alpha\right] \tag{16}$$

In order to determine the asymptote being parallel to the ordinate, the denominator is equalized to zero, and the value x is solved:

$$x = \frac{t\cos\alpha - s\cot\alpha}{2\sin\alpha} \tag{17}$$

Huerta noted the "discontinuity of the curvature of the line of thrust" at the common point B between two plate-bandes [1, p. 247]. With respect to the adopted coordinate system, the tangent of horizontal line through the point B equals tanα. In order to determine the direction of the tangent line to the thrust line at the point B, the value x in Eq. (5) is substituted by zero. Hence the inclination of the tangent line is given by:

$$\frac{s\tan\alpha}{s - t\sin\alpha} \tag{18}$$

The value $t\sin\alpha$ is smaller than s, so $s/(s-t\sin\alpha)$ is always greater than one. Therefore, the ratio given by Eq. (18) is greater than tanα, so that the tangent at the crown is inclined rather than horizontal, and the thrust line goes slightly upwards from the point B, as shown in Fig. 5b. Hence, the thrust line as a whole, within the both sides of the structure, has the cusp at their common point, as one can see in Fig. 5c. Since the horizontal thrust is the only one acting at the crown, this might appear somewhat unnatural, but it can be attributed to the part (weight) of the structure that is above the crown.

4. CONCLUSION

Thrust line theory enables the exact examination of the stability of two inclined brick plate-bandes leaning against each other. This apparently simple problem, regarding the determination of the maximum number of bricks laid each on other and inclined at some angle, has not been solved until now. Analytical expressions of thrust line, representing a hyperbola, as well as the expression for the minimum thickness to length ratio are derived. Accordingly, it has been shown that Young's solution turns out to be incorrect, since even 20 rather than 15 bricks, inclined at the angle of 60°, would stand. Conducted analysis can be used as the base for the analyses of similar types of triangular and false arches.

ACKNOWLEDGEMENTS

The paper was done within the Project No. TR36042 supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

REFERENCES

[1] Huerta, S.: Wedges and plate-bandes: mechanical theories after De la Hire. In: Gargiani R. (ed.) *L'architrave, le plancher, la plate-forme. Nouvelle Histoire de la construction.* Presses polytechniques et universitaires romandes, Lausanne, **2012**, pp. 405–435.

[2] Young, T. [signed APSOPHUS]: Remarks on the Structure of covered Ways, independent of the Principle of the Arch in Equilibrium, and on the Best Forms for Arches in Buildings. *A Journal of Natural Philosophy, Chemistry and the Arts*, **1807**, 18, pp. 241–250 and pl. VII.

[3] Barlow, W. H.: On the Existence (practically) of the line of equal Horizontal Thrust in Arches, and the mode of determining it by Geometrical Construction. *Minutes and Proceedings of the Institution of Civil Engineers*, **1846**, 5, pp. 162–182.

[4] Couplet, P.: *De la poussée des voûtes*. Histoire de l'Académie Royale des Sciences, Paris, **1730**, pp. 117–141.

[5] Milankovitch, M.: Theorie der Druckkurven. *Zeitschrift für Mathematik und Physik,* **1907**, 55, pp. 1–27.

[6] Alexakis, H., Makris, N.: Minimum thickness of elliptical masonry arches. *Acta Mechanica*, **2013**, 224, pp. 2977–2991.

[7] Nikolić, D.: Thrust line analysis and the minimum thickness of pointed masonry arches. *Acta Mechanica*, **2017**. doi:10.1007/s00707-017-1823-6

[8] Heyman, J.: *Stone Skeleton: Structural Engineering of Masonry Architecture*. Cambridge university press, Cambridge, **1997**.

БЕЛЕШКА О АНАЛИЗИ РАВНОТЕЖЕ НАГНУТОГ ПРАВОГ ЛУКА

Резиме: *Томас Јанг је 1807. године објавио чланак о различитим конструкцијским облицима лажних лукова. Један од разматраних проблема била је стабилност двају правих лукова од опеке нагнутих један на други. У овом раду је, применом геометријског приступа у статичком испитивању, овај проблем аналитички обрађен, те је изведен израз за потпорну линију. Уз то је испитано гранично стање равнотеже, те је одређен минимални однос дебљине и дужине у функцији нагибног угла. У складу с тим, испоставља се да је Јангово решење нетачно, будући да је, уместо петнаест, чак двадесет опека стабилно у правом луку нагнутом под углом од шездесет степени.*

Кључне речи: *потпорна линија, нагнути прави лук, статичка анализа, Томас Јанг*